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RANGE OF THE GRADIENT OF A SMOOTH BUMP FUNCTION IN FINITE DIMENSIONS

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ABSTRACT. This paper proves the semi-closedness of the range of the gradient for sufficiently smooth bumps in the Euclidean space.

Let $\mathbb{R}^N$ be the Euclidean space of dimension N. A *bump* on R^N is a function from $\mathbb{R}^N$ into $\mathbb{R}$ with a bounded nonempty support. The aim of this short paper is to partially answer an open question suggested by Borwein, Fabian, Kortezov and Loewen in [1]. Let $f : \mathbb{R}^N \to \mathbb{R}$ be a C^1-smooth bump function; does $f'(\mathbb{R}^N)$ equal the closure of its interior? We are not able to provide an answer, but we can prove the following result.

Theorem 0.1. *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a C^{N+1}-smooth bump. Then $f'(\mathbb{R}^N)$ is the closure of its interior.*

We do not know if the hypothesis on the regularity of the bump f is optimal in our theorem when $N \geq 3$. However, the result can be improved for $N = 2$; Gaspari [3] proved by specific two-dimensional arguments that the conclusion holds if the bump is only assumed to be C^2-smooth on the plane. Again we cannot say if we need the bump function to be C^2 for $N = 2$. We proceed now to prove our theorem.

1. Proof of Theorem 0.1

For the sequel, we set $F := f' = \nabla f$. Moreover, since the theorem is obvious for $N = 1$ we will assume that $N \geq 2$. The proof is based on a refinement of Sard's Theorem that can be found in Federer [2]. Let us denote by B_k, C_k ($k \in \{0, \cdots, N\}$) the sets defined as follows:

$$B_k := \{x \in \mathbb{R}^N : \operatorname{rank} DF(x) \leq k\},$$
$$C_k := \{x \in \mathbb{R}^N : \operatorname{rank} DF(x) = k\}.$$

Of course $C_k \subseteq B_k$ and $B_N = \mathbb{R}^N$. Theorem 3.4.3 in [2] says that if the function F is C^N-smooth, then for all $k = 0, \cdots, N-1$,

$$\mathcal{H}^{k+1}\left(F(B_k)\right) = \mathcal{H}^{k+1}\left(F(C_k)\right) = 0, \tag{1.1}$$

where $\mathcal{H}^{k+1}$ denotes the $(k+1)$-dimensional Hausdorff measure.

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Fix $\bar{x}$ in $\mathbb{R}^N$ and let us prove that $F(\bar{x})$ belongs to the closure of $\operatorname{int}(F(\mathbb{R}^N))$. Since it is well known that $0 \in \operatorname{int}(F(\mathbb{R}^N))$ (see Wang [6]), we can assume that $F(\bar{x}) \neq 0$. Our proof begins with the following lemma.

Lemma 1.1. *There exists a neighbourhood $\mathcal{V}$ of $F(\bar{x})$ relative to $F(\mathbb{R}^N)$ and an integer $\bar{k} \in \{1, \cdots, N\}$ such that for any $x \in F^{-1}(\mathcal{V}), \operatorname{rank} DF(x) \leq \bar{k}$ and there exists a sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{V}$ which converges to $F(\bar{x})$ such that*

$$F^{-1}(v_n) \subseteq \operatorname{int}(C_{\bar{k}}). \tag{1.2}$$

Proof. Let us fix V an open neighbourhood of $F(\bar{x})$ relative to $F(\mathbb{R}^N)$ and denote by k_0 the max of the k's in $\{0, 1, \cdots, n\}$ which satisfy $V \cap F(C_k) \neq \emptyset$.

First of all we remark that $k_0 > 0$. As a matter of fact, suppose that for any $k \geq 1, V \cap F(C_k) = \emptyset$, that is, for any y in $F^{-1}(V), \operatorname{rank} DF(y) = 0$. Since $F^{-1}(V)$ is open this implies that F is constant on $F^{-1}(V)$ and hence that $F(\bar{x})$ is isolated in $F(\mathbb{R}^N)$. So, we get a contradiction by arc-connectedness of $F(\mathbb{R}^N)$ (and since $F(\bar{x}) \neq 0$ and $0 \in F(\mathbb{R}^n)$). Consequently, we deduce that there exists $y \in \mathbb{R}^N$ such that $F(y) \in V$ and $\operatorname{rank} DF(y) = k_0 > 0$. Furthermore for all $z \in F^{-1}(V), \operatorname{rank} DF(z) \leq k_0$. Hence by lower semicontinuity of $z \mapsto \operatorname{rank} DF(z)$, this implies that $\operatorname{rank} DF$ is constant in a neighbourhood of y (because $\{z : \operatorname{rank} DF(z) \geq k_0\}$ is open). Therefore, by the rank theorem (see Rudin [4, Theorem 9.20]), V has the structure of a k_0-dimensional manifold near $F(y)$, and hence $\mathcal{H}^{k_0}(V) > 0$. Thus by (1.1), $V \setminus F(B_{k_0-1})$ is nonempty. We conclude that for any v in the latter set,

$$F(z) = v \Longrightarrow \operatorname{rank} DF(z) = k_0;$$

in addition z has a neighbourhood on which $\operatorname{rank} DF \leq \bar{k}$ by choice of $\bar{k}$, and the set where $\operatorname{rank} DF \geq \bar{k}$ is open. Consequently such a v satisfies $F^{-1}(v) \subseteq \operatorname{int}(C_{\bar{k}})$. Repeating this argument with a decreasing sequence on neighbourhoods, we get a decreasing sequence of k_0-values in $\{1, \cdots, n\}$ which has to be stationary. Hence the proof is easy to complete. □

We now claim the following lemma.

Lemma 1.2. *The constant of Lemma 1.1 satisfies $\bar{k} = N$.*

Proof. Let us remark that since $F = f' = \nabla f$, the Jacobian of F at any point y in $\mathbb{R}^N$ is actually the Hessian of the function f. We argue by contradiction and so we assume that $\bar{k} < N$.

By the previous remark, for any $y \in \mathbb{R}^N$, $DF(y)$ is a symmetric matrix, the nontrivial vector subspaces $\operatorname{Ker} DF(y)$ and $\operatorname{Im} DF(y)$ are orthogonal, and $DF(y)$ induces an automorphism on $\operatorname{Im} DF(y)$. Let us fix $n \in \mathbb{N}$. By Lemma 1.1 and by the constant rank theorem (see for instance Spivak [5] page 65) we deduce that $M_n := \{y : F(y) = v_n\}$ is a submanifold of $\mathbb{R}^N$ of dimension $N - k$ and at least C^2-smooth (since F is C^N-smooth and $N \geq 2$). Furthermore since f is a bump, M_n is a compact submanifold.

Now since M_n is a C^2 submanifold of $\mathbb{R}^N$ there exists an open tubular neighbourhood $\mathcal{U} \subset \mathcal{V}$ of M_n and a C^2-smooth function $r : \mathcal{U} \to M_n$ which is the projection on the set M_n such that for any $x \in \mathcal{U}, x - r(x) \in N_{r(x)}M_n$, where for any $p \in M_n$, N_pM_n denotes the normal space of M_n at p. In addition, from the properties of the constant $\bar{k}$, by reducing $\mathcal{U}$ if necessary, we can assume that for any $x \in \mathcal{U}, \operatorname{rank} DF(x) = \bar{k}$. We set the following function on the neighbourhood $\mathcal{U}$:

$$\begin{aligned} \Phi : \mathcal{U} &\to \mathbb{R}^N, \\ x &\mapsto DF(r(x))(x - r(x)). \end{aligned}$$

We now need the following result.

Lemma 1.3. *If M_n is a compact C^2 submanifold of $\mathbb{R}^N$, then for all ξ in the unit sphere $\mathbb{S}^{N-1}$, there exists $p \in M_n$ such that $\xi \in N_p M_n$.*

Proof. Consider for any $l \in \mathbb{N}$, $p_l := \operatorname{proj}_{M_n}(l\xi)$, where $\operatorname{proj}_{M_n}(\cdot)$ denotes the projection map on the closed set M_n. Since the submanifold M_n is C^2, the vector $\frac{l\xi - p_l}{\|l\xi - P_l\|}$ belongs to $N_{p_l} M_n$. Moreover by compactness of M_n we can assume that $p_l \to \bar{p}$ when l tends to infinity. Now since the sequence $(p_l)_{l \in \mathbb{N}}$ is bounded, we have that $\lim_{l\to\infty} \frac{l\xi - p_l}{\|l\xi - p_l\|} = \xi$. By continuity of the normal bundle NM_n, we easily conclude that $\xi \in N_{\bar{p}} M_n$. □

Returning to the proof of Lemma 1.2, Lemma 1.3 immediately implies that for all $\xi \in \mathbb{S}^{N-1}$, there exists $p \in M_n$ and $v \in N_p M_n$ such that $v = \xi$. Furthermore the map $DF(p)$ is an automorphism on $N_p M_n$, hence there exists $w \in N_p M_n$ such that $DF(p)(w) = v$. We conclude that for any t small enough (s.t. $p + tw \in \mathcal{U}$), $DF(p)(tw) = t\xi$ and hence that $\Phi(p + tw) = t\xi$. Furthermore since M_n is compact and since the map $p \mapsto [DF(p)_{|N_p M_n}]^{-1}$ is continuous on M_n, we deduce that $\|w\|$ is bounded above. Hence by compactness on M_n, we get that for some $t_0 > 0$ the ball $B(0, t_0)$ is included in $\Phi(\mathcal{U})$; hence $\Phi(\mathcal{U})$ has a nonempty interior. Therefore (since the function Φ is smooth enough) Sard's Theorem gives us the existence of regular values of Φ in $\mathbb{R}^N$. So there exists $\bar{y} \in \mathcal{U}$ such that $\operatorname{rank} D\Phi(\bar{y}) = N$. Consequently there exists $\rho > 0$ such that the map Φ is one-to-one on $\mathcal{W} = B(\bar{y}, \rho)$ (the ball centered at $\bar{y}$ with radius ρ).

For any $l \in \mathbb{N}^*$, we set $y_l := r(\bar{y}) + \frac{1}{l}(\bar{y} - r(\bar{y}))$. The constant rank theorem implies that for any l the set $V_l := \{y \in \mathcal{U} : F(y) = F(y_l)\}$ is a submanifold of $\mathcal{U}$ of dimension $N - \bar{k}$. (Of course V_l might be noncompact in $\mathcal{U}$, *i.e.* $\overline{V_l}$ not included in $\mathcal{U}$.) On the other hand, by Lipschitz continuity of $DF(\cdot)$ and since $N - \bar{k} > 0$, there exists a neighbourhood $\mathcal{Y}$ of the segment $[\bar{y}, r(\bar{y})]$ in $\operatorname{co}\{\mathcal{W} \cup r(\mathcal{W})\}$ and a Lipschitz continuous map $X : \mathcal{Y} \to \mathbb{R}^N$ such that for any $x \in \mathcal{Y}$,

$$X(x) \in \ker DF(x) \text{ and } \|X(x)\| = 1.$$

If we denote by $\theta_X(y, \tau)$ the local flow of the vector field X on $\mathcal{Y}$, we get that for any τ small enough, $\theta_X(y_l, \tau) \in V_l$. On the other hand, Gronwall's Lemma easily yields the following (we omit the proof):

Lemma 1.4. *There exist two positive constants K, μ such that for any $l \in \mathbb{N}^*$ and for any $\tau \leq \mu$, we have*

$$\theta_X(y_l, \tau) \in \operatorname{co}\left\{ B\left(\bar{y}, \frac{\rho}{2}\right) \cup r\left(B\left(\bar{y}, \frac{\rho}{2}\right)\right)\right\}, \tag{1.3}$$

$$\frac{\|\theta_X(y_l, \tau) - r(\theta_X(y_l, \tau))\|}{\|y_l - r(y_l)\|} \in [e^{-K\tau}, e^{K\tau}]. \tag{1.4}$$

We now conclude the proof of Lemma 1.2. We set for any $l \in \mathbb{N} \setminus \{0\}$, $z_l := \theta_X(y_l, \mu)$. First remark that if μ is small enough, then we have (recall that $\|X\| = 1$)

$$\begin{aligned} & \langle X(\theta_X(y_l, s)), X(y_l)\rangle \geq \frac{1}{2} \\ \Longrightarrow \quad & \langle \int_0^\mu X(\theta_X(y_l, s))ds, X(y_l)\rangle \geq \frac{\mu}{2} \\ \Longrightarrow \quad & \|z_l - y_l\| \geq \frac{\mu}{2}. \end{aligned} \tag{1.5}$$

By considering a converging subsequence of $(z_l)_{l\in\mathbb{N}^*}$ if necessary we can compute

$$\begin{aligned} \lim_{l\to\infty} \frac{F(y_l) - F(r(y_l))}{\|z_l - r(z_l)\|} &= \lim_{l\to\infty} \frac{F(z_l) - F(r(z_l))}{\|z_l - r(z_l)\|} \\ &= \lim_{l\to\infty} DF(r(z_l)) \left(\frac{z_l - r(z_l)}{\|z_l - r(z_l)\|} \right) \\ &= DF(\bar{z})(\bar{\zeta}), \end{aligned}$$

where $\lim_{l\to\infty} z_l = \bar{z} = r(\bar{z}) \in M_n$ and $\lim_{l\to\infty} \frac{z_l - r(z_l)}{\|z_l - r(z_l)\|} = \bar{\zeta} \in N_{\bar{z}} M_n$. We deduce that

$$\begin{aligned} DF(r(\bar{y}))(\bar{y} - r(\bar{y})) &= \lim_{l\to\infty} l\,(F(y_l) - F(r(y_l))) \\ &= \lim_{l\to\infty} l\|z_l - r(z_l)\| \frac{F(y_l) - F(r(y_l))}{\|z_l - r(z_l)\|} \\ &= c\|\bar{y} - r(\bar{y})\| DF(\bar{z})(\bar{\zeta}) \\ &= DF(\bar{z})(c\|\bar{y} - r(\bar{y})\|\bar{\zeta}), \end{aligned}$$

with $c = \lim_{l\to\infty} \frac{\|z_l - r(z_l)\|}{\|y_l - r(y_l)\|}$.

The computations prove that $\Phi(\bar{y}) = \Phi(\bar{z} + c\|\bar{y} - r(\bar{y})\|\bar{\zeta})$. Furthermore by (1.3) and (1.5), $\bar{z}$ belongs to $r(\mathcal{W})$ and $\|\bar{z} - r(\bar{y})\| > 0$. Consequently since Φ is injective on $\mathcal{W}$, it remains to prove that $\bar{z} + c\|\bar{y} - r(\bar{y})\|\bar{\zeta}$ is in $\mathcal{W}$ to get a contradiction. By (1.4) taking μ smaller if necessary, we get the result of Lemma 1.2. □

The proof of Theorem 0.1 is now easy. Since $\bar{k} = N$, for any $n \in \mathbb{N}$ the different values v_n of Lemma 1.1 belong to the interior of $f'(\mathbb{R}^N)$ and moreover the sequence $(v_n)_{n\in\mathbb{N}}$ converges to $F(\bar{x})$. This proves the theorem.

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